Functions of several variables

(M.Sc. (MATHEMATICS), Paper - VI)

(Real Analysis - II)

Lecture - 03

Sudip Kumar
Assistant Professor
Department of Mathematics
S. Sinha College,
Aurangabad (Bihar)

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functions of two variables

Example ③ Show that

$$\lim_{(x,y)\to(0,0)} xy\,\frac{x^2-y^2}{x^2+y^2} = 0$$

Solution: → Let $f(x,y) = xy\,\frac{x^2-y^2}{x^2+y^2}$

put $x = r\cos\theta,\ y = r\sin\theta$

$$\therefore\ |f(x,y) - 0| = \left| xy\,\frac{x^2-y^2}{x^2+y^2} \right|$$

$$= \left| r^2 \sin\theta\cos\theta\,\frac{r^2\cos^2\theta - r^2\sin^2\theta}{r^2\cos^2\theta + r^2\sin^2\theta} \right|$$

$$= \left| r^2 \sin\theta\cos\theta\,\cos 2\theta \right|$$

$$= \left| \frac{r^2}{4} \cdot 2\,(2\sin\theta\cos\theta)\,\cos 2\theta \right|$$

$$= \left| \frac{r^2}{4} \sin 4\theta \right| \le \frac{r^2}{4} \qquad (\because |\sin 4\theta| \le 1)$$

i.e; $|f(x,y) - 0| \le \frac{r^2}{4} = \frac{x^2+y^2}{4} < \epsilon$

If $\frac{x^2}{4} < \frac{\epsilon}{2}$ & $\frac{y^2}{4} < \frac{\epsilon}{2}$

or if $|x-0| < \sqrt{2\epsilon} = \delta,\ |y-0| < \sqrt{2\epsilon} = \delta$

Thus for any $\epsilon > 0,\ \exists\ \delta(\epsilon) > 0$ s.t.

$$\left| xy\,\frac{x^2-y^2}{x^2+y^2} - 0 \right| < \epsilon, \text{ whenever } |x| < \delta \ \&\ |y| < \delta$$

$$\therefore \lim_{(x,y)\to(0,0)} f(x,y) = 0$$

proved.

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Repeated limits :→ Let $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ defined in some neighbourhood of (a,b), then the limit

$$\lim_{y \to b} f(x,y)$$

if exists, is a function of x, say $\phi(x)$.

If then the limit $\lim_{x \to a} \phi(x)$ exists & is equal to λ, we write

$$\lim_{x \to a} \lim_{y \to b} f(x,y) = \lambda$$

and say that λ is a repeated limit of f as $y \to b$, $x \to a$.

If we change the order of taking the limits, we get the other repeated limit

$$\lim_{y \to b} \lim_{x \to a} f(x,y) = \lambda' \text{ (say)}$$

When first $x \to a$, then $y \to b$.

Remark :→ If simultaneous limit exists, then these two repeated if exist are necessarily equal. but not conversely.

Example :→ Let $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ defined as

$$f(x,y) = \frac{xy}{x^2+y^2}. \text{ then}$$

the repeated limits

$$\lim_{y \to 0} \lim_{x \to 0} f(x,y) = \lim_{y \to 0} 0 = 0$$

and $\lim_{x\to 0} \lim_{y\to 0} f(x,y) = \lim_{x\to 0} 0 = 0$

i.e; repeated limit exists and they are equal to 0.

The simultaneous limit does not exist:

Let ~~for~~ $(x,y) \to (0,0)$, along the path $y = mx$ & $x \to 0$

$$\therefore \lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$

$$= \lim_{\substack{y=mx \\ x\to 0}} \frac{mx^2}{x^2(1+m^2)}$$

$$= \lim_{x\to 0} \frac{m}{1+m^2} = \frac{m}{1+m^2}$$

Which is different for different values of m.

$\therefore \lim_{(x,y)\to(0,0)} f(x,y)$ does not exists.

Example ②:→ Show that the limit exists at the origin but the repeated limit ~~to~~ ~~cannot~~ donot, Where

$$f(x,y) = \begin{cases} x\sin\left(\frac{1}{y}\right) + y\sin\left(\frac{1}{x}\right), & xy \neq 0 \\ 0, & xy = 0 \end{cases}$$

Solution:→

Here $\lim_{y\to 0} f(x,y)$ and $\lim_{x\to 0} f(x,y)$ does not exists.

Therefor $\lim_{x\to 0} \lim_{y\to 0} f(x,y)$; $\lim_{y\to 0} \lim_{x\to 0} f(x,y)$ does not exists.

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Again, $|f(x,y) - 0|$

$$\left| x\sin\frac{1}{y} + y\sin\frac{1}{x} - 0 \right| < |x| + |y|$$

$$\leq 2\sqrt{x^2+y^2} < \epsilon$$

Here $x^2 < \frac{\epsilon^2}{4}$, $y^2 < \frac{\epsilon^2}{4}$

$\therefore |x-0| < \frac{\epsilon}{2} = \delta$, $|y-0| < \frac{\epsilon}{2} = \delta$

Thus for any $\epsilon > 0$, $\exists\ \delta > 0$ s.t.

$|f(x,y) - 0| < \epsilon$ when $|x| < \delta$, $|y| < \delta$

$$\therefore \lim_{(x,y)\to(0,0)} f(x,y) = 0$$

Hence simultaneous limit exist but repeated limit does not exists.